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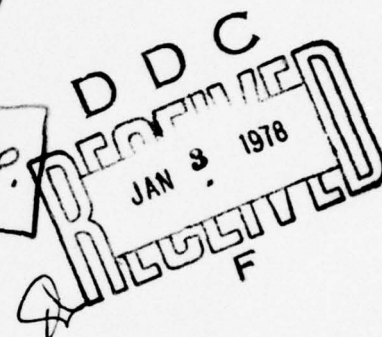
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AN IMPROVED FINITE ELEMENT FORMULATION DERIVED FROM  
THE METHOD OF WEIGHTED RESIDUALS,

C.A.J. Fletcher

11 Jun 77



## SUMMARY

By replacing the residual with its least-squares fit an improvement has been made to the Method of Weighted Residuals. For finite element applications the least-squares fit is made on an element basis. The improved formulation is illustrated by a number of examples drawn mainly from fluid-flow problems. The new formulation is closely related to the use of reduced integration and detailed comparisons with both exact and reduced integration are presented. The present formulation produces superior results to the use of reduced integration for some problems and formulations particularly when used with quadratic triangular elements.

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## 1. INTRODUCTION

The main limitations on the greater use of three-dimensional finite element calculations are excessive CPU time and excessive space requirements to achieve an acceptable accuracy. If it is possible to obtain more accurate solutions from coarse grids the benefits for three-dimensional calculations are obvious. For certain problems reduced numerical integration(refs.1,2) produces more accurate solutions than exact numerical integration. The motivation for the present work has come, in part, from a desire to explain why reduced integration often produces more accurate solutions. The novel finite element formulation presented here has arisen through isolating and generalising the main feature of reduced integration that is responsible for its success.

## 2. REDUCED INTEGRATION

With the widespread usage of isoparametric elements, numerical integration, rather than analytic integration, has become almost mandatory. Gauss quadrature formulae have proved to be very efficient for numerical integration over rectangular elements; an  $n^{\text{th}}$  order Gauss quadrature formula will ensure that polynomial integrands up to order  $2n-1$  are integrated exactly. Reducing the order of integration introduces an error into the evaluation of the integrals but it often produces a more accurate final solution.

Previous applications of reduced integration have been to plate and shell problems(refs.1,2) to elasto-static problems(ref.3), to plastic flow problems(ref.4), to fracture analysis(ref.5), to convective transport problems(ref.6), and to incompressible, inviscid flow(ref.7). Reduced integration has been compared with other methods of directly approximating the strain field in structural applications of the finite element method by Argyris and William(ref.8). A description of the general features of reduced integration has been given by Zienkiewicz(ref.9).

An appreciation of how reduced integration works may be obtained by deriving the stiffness matrix for a Galerkin finite element formulation in which the integration is performed numerically. Suppose a solution for  $q$  is sought in a two-dimensional domain. Once an analytic representation, of the form

$$q(x,y) = \sum_j N_j(x,y) \cdot \bar{q}_j, \quad (1)$$

is substituted into the governing equation,  $L(q) = 0$ , a residual is created,

$$R(\bar{q}_j, x, y) = L(\bar{q}_j, x, y). \quad (2)$$

The Galerkin formulation requires that

$$\iint N_i(x,y) \cdot R(\bar{q}_j, x, y) \cdot dx dy = 0, \quad i = 1, n. \quad (3)$$

$N_i$  is the shape function corresponding to the  $i^{\text{th}}$  node. Clearly repeated application of equation (3) with  $n$  different weight functions,  $N_i$ , will produce enough independent, algebraic equations to solve for the  $n$  nodal unknowns,  $\bar{q}_j$ . If the integration in equation (3) is performed numerically the result is

$$\sum_{k=1}^M w_k \cdot N_i(x_k, y_k) \cdot R_k(\bar{q}_j, x_k, y_k) = 0, i = 1, n. \quad (4)$$

$w_k$  is a weight, determined by the quadrature formula, attached to the  $k^{\text{th}}$  function evaluation point.  $R_k$  is the value of the residual at  $x_k, y_k$  and  $M$  is the total number of function evaluation points in the domain.

The residual  $R_k$  could depend on any of the nodal unknowns,  $\bar{q}_j$ . Thus, for linear problems,

$$R_k = \sum_{j=1}^L R_{k,j}(x_k, y_k) \cdot \bar{q}_j. \quad (5)$$

Substitution of equation (5) into equation (4) produces the result

$$\sum_{k=1}^M w_k \cdot N_i(x_k, y_k) \cdot \sum_{j=1}^L R_{k,j}(x_k, y_k) \cdot \bar{q}_j = 0, i = 1, n. \quad (6)$$

In equations (5) and (6)  $L$  represents the total number of nodal parameters,  $\bar{q}_j$ , whereas  $n$  represents the number of nodal parameters,  $\bar{q}_j$ , that are unknown.

Equation (5) represents one linear relationship between the  $n$  unknowns,  $\bar{q}_j$ . Each additional function evaluation point  $(x_k, y_k)$  introduces another linear relationship between the  $\bar{q}_j$ 's. At least  $n$  evaluation points will be required to ensure that enough independent linear relationships are available for solution for  $\bar{q}_j$ . Directly setting  $R_k = 0$  for  $n$  values of  $(x_k, y_k)$  would produce a solution by the collocation method, which is a subgroup of the method of weighted residuals. The Galerkin formulation, equation (6), effectively allows all the function evaluation points,  $M$ , to contribute to the solution. For a solution to be possible with a Galerkin formulation it is necessary that

$$M \geq n. \quad (7)$$

In practice if the quadrature formula is chosen so that the numerical integration is carried out exactly  $M$  may exceed  $n$  by a factor of 2 or 3(ref.7). The excess evaluations of the residual arise from the need for the solution to be constrained by the local analytic representation given by equation (1). The use of reduced numerical integration implies a smaller number of evaluations,  $M$ , of the integrand. This relaxes some of the constraints on the solution at



the expense of introducing additional errors by not performing the integration exactly. The empirical evidence(refs.1,9) clearly indicates that the gains associated with relaxing the constraints far outweigh the losses associated with executing the numerical integration less accurately.

It appears from equation (7) that the maximum reduction of the order of the quadrature scheme will occur when  $M = n$ . However the minimum value for  $M$  is more likely to be determined by the requirement of convergence of the formulation. For an isoparametric formulation, Irons(ref.10) gives the minimum order of integration as that which permits exact integration of the element area (in two dimensions).

If the order of summation in equation (6) is reversed the familiar stiffness equation is obtained i.e.

$$\sum_{j=1}^L K_{ij} \cdot \bar{q}_j = 0, \quad i = 1, n \quad (8)$$

where

$$K_{ij} = \sum_{k=1}^M w_k \cdot N_i(x_k, y_k) \cdot R_{kj}(x_k, y_k).$$

### 3. PRESENT FORMULATION

The Method of Weighted Residuals(ref.11) has proved to be a useful framework for relating apparently unconnected methods both inside and outside the finite element formulation(ref.12). The formulation to be presented here grew out of an attempt to relate reduced integration to the framework that supports the method of weighted residuals. Just as the method of weighted residuals includes a broader class of methods than finite element methods so the current formulation is also applicable outside the finite element area of application.

The method of weighted residuals (MWR) is a suitable technique to use when a numerical solution is sought for

$$L(v) = 0, \quad (9)$$

where  $L$  is a differential operator. An approximate solution,  $u$ , is sought within some domain  $D$  and subject to boundary conditions on  $S$ , the boundary of  $D$ . For steady problems, MWR requires the introduction of a trial solution of the form (in two dimensions)

$$u(x, y) = \sum_{k=1}^N a_k \cdot \phi_k(x, y). \quad (10)$$



Substitution of equation (10) into the differential equation, (9), produces some residual,

$$R = L(u)$$

$$= \sum_{k=1}^N a_k \cdot L(\phi_k). \quad (11)$$

If the approximate solution,  $u$ , given by equation (10) contains the exact solution,  $v$ , the residual,  $R$ , will be zero throughout the domain,  $D$ . MWR approximates this situation by requiring that the integral over the domain of the weighted residual is zero. Thus

$$\iint W_i \cdot R \cdot dx \, dy = 0. \quad (12)$$

By repeated evaluation of equation (12), with different weight functions,  $W_i$ , enough algebraic relations are established to evaluate the unknown coefficients,  $a_k$ , in equation (10). The present improvement to MWR consists of replacing  $R$  by its least-squares fit over the domain. Thus

$$\iint W_i \cdot R_{1s} \cdot dx \, dy = 0 \quad (13)$$

is used instead of equation (12) to obtain the algebraic relations between the unknown coefficients,  $a_k$ .  $R_{1s}$  is obtained from

$$\iint (R - R_{1s})^2 \, dx \, dy = \text{minimum}. \quad (14)$$

In one dimension with  $\phi_k$  as polynomials in  $x$ , equation (14) can be differentiated to give

$$\int x^k \cdot R \cdot dx = \int x^k \cdot R_{1s} \cdot dx, \quad k = 0 \dots M-1, \quad (15)$$

where  $M$  is the order (in  $x$ ), of the residual,  $R$ . If the Galerkin method is chosen as the example of MWR equation (13) has the form

$$\int x^i \cdot R_{1s} \cdot dx = 0. \quad (16)$$

Thus by setting equation (15) equal to zero the replacement of the residual by its least-squares fit may be interpreted, for this one-dimensional example, as a generalised Galerkin method(ref.12) in which the normal Galerkin weighting function,  $x^i$  is replaced by  $x^{i-1}$ . This idea will be illustrated by the first example.

Since a large number of efficient finite element formulations already exist some justification for introducing another would appear warranted. At the present time the justification is entirely pragmatic : for the problems considered the new formulation has produced more efficient solutions than a conventional Galerkin finite element formulation and has demonstrated a wider range of applicability than reduced integration.

#### 4. EXAMPLES

The current finite element formulation will be developed by considering a number of examples of increasing complexity which will illustrate different aspects of the formulation. To permit an easier examination of how MWR works, and how it can be improved, the first example will be solved by the traditional MWR formulation, i.e. the approximating functions will span the whole domain rather than being restricted to a local region as in the finite element method.

##### 4.1 Classical MWR applied to $dy/dx - y = 0$

A solution is sought, for the equation  $dy/dx - y = 0$  and boundary condition  $y = 1$  at  $x = 0$ , in the domain  $0 \leq x \leq 1$ . It is of historical interest that this example was used by Duncan(ref.13) to illustrate the traditional Galerkin method.

The following approximate solution is introduced

$$y_a = 1 + a_1 \cdot x + a_2 \cdot x^2, \quad (17)$$

and the coefficients  $a_1$  and  $a_2$  are to be determined. It may be noted that equation (17) satisfies the boundary condition exactly. Substitution of the approximate solution, equation (17), into the governing equation produces a residual,  $R$ , given by

$$R = -1 + a_1 \cdot (1 - x) + a_2 \cdot (2x - x^2). \quad (18)$$

Since  $R$  is quadratic in  $x$  it is impossible for  $a_1$  and  $a_2$  to be chosen so that  $R$  is identically zero, unless the choice of the approximate solution, equation (17), happens to contain the exact solution.

In order to determine  $a_1$  and  $a_2$  the integral, over the domain, of the weighted residual is set equal to zero. Thus

$$\int_0^1 W_i \cdot R \cdot dx = 0, \quad i = 1, 2. \quad (19)$$

If  $W_i = x^i$ , i.e. the same as the analytic function in equation (1), the traditional Galerkin method is obtained. Evaluation of equation (19) leads to the following algebraic equations for  $a_1$  and  $a_2$ ,

$$\frac{1}{6} \cdot a_1 + \frac{5}{12} \cdot a_2 = \frac{1}{2}$$

$$\frac{1}{12} \cdot a_1 + \frac{3}{10} \cdot a_2 = \frac{1}{3} \quad (20)$$

From equations (20),  $a_1 = 8/11$  and  $a_2 = 10/11$ . The solution for  $y_a$  is shown in Table 1 as  $y_{Gal}$ .

TABLE 1. MWR SOLUTIONS FOR  $dy/dx - y = 0$

x	Approximate solution			Exact solution $y = e^x$
	$y_{Gal}$	$y_{res. \text{ l.s.}}$	$y_{class. \text{ l.s.}}$	
0	1.0000	1.0000	1.0000	1.0000
0.2	1.1818	1.2057	1.2072	1.2214
0.4	1.4364	1.4800	1.4819	1.4918
0.6	1.7636	1.8288	1.8241	1.8221
0.8	2.1636	2.2343	2.2337	2.2255
1.0	2.6364	2.7143	2.7108	2.7183
$\int_0^1 R^2 dx$	0.00826	0.00408	0.00402	0.

Substitution of the solutions for  $a_1$  and  $a_2$  into equation (18) will produce a residual that is non-zero. Intuitively the closer  $R$  is to zero the closer  $y_a$  should be to the exact solution. The conventional way of obtaining an improvement in the solution  $y_a$  is to allow more unknown coefficients  $a_j$  in equation (17). In the limit of having an infinite number of unknown coefficients,  $a_j$ , the residual can be made identically zero and the approximate solution,  $y_a$ , coincides with the exact solution. Increasing the number of unknown coefficients,  $a_j$ , is not a very practical technique because of difficulties in solving the algebraic equations when the number of unknowns is large (ref. 12).

A technique for reducing the size of  $R$ , without increasing the number of unknown coefficients,  $a_j$ , may be obtained by approximating the residual, in some sense, by a lower order analytic function. Thus if equation (18) is fitted, in the least-squares sense, by a function linear in  $x$ , the result is

$$R_{l.s.} = -1 + a_1 (1 - x) + a_2 \left(x + \frac{1}{6}\right). \quad (21)$$



Now it is possible to ensure that  $R_{1.s.}$  is identically zero by setting the coefficients of  $x^0$  and  $x^1$  equal to zero. This produces the result  $a_1 = 6/7$  and  $a_2 = 6/7$ . The same result for  $a_1$  and  $a_2$  would result from applying any of the methods of weighted residuals in the form of equation (13).

The corresponding solution for  $y_a$  is plotted in Table 1 as  $y_{res. 1.s.}$  and it is apparent that it is considerably closer to the exact solution than that produced by the conventional Galerkin method.

The success of the least-squares fit of the residual is presumably due to the fact that  $R_{1.s.}$  has the same global character as  $R$  but the lower order of analytic functions present has reduced the constraints on the unknown coefficients, so that they may be chosen so that  $R_{1.s.}$  is identically zero.

The most accurate solution might be expected from the classical least-squares formulation, since this is obtained by requiring

$$\int_0^1 R^2 dx = \min. \quad (22)$$

In terms of the method of weighted residuals this is equivalent to setting  $W_i = \partial R / \partial a_i$ . For the above problem  $W_i = (1 - x)$  and  $(2x - x^2)$ . The solution for the classical least-squares formulation is also shown in Table 1 and it is apparent from both individual values and the evaluation

of  $\int_0^1 R^2 dx$  that the solution, after a least-squares fit of the residual, is very close to that obtained from the classical least-squares formulation. Equation (21) can be written in the form

$$R_{1.s.} = b_0 + b_1 \cdot x \quad (23)$$

and the solution is obtained by requiring that  $b_0, b_1 = 0$ . These conditions can be substituted into the equations that are used to calculate  $b_0$  and  $b_1$  i.e.

$$\int_0^1 R_{1.s.} x^j dx = \int_0^1 R \cdot x^j dx = 0, j = 0, 1. \quad (24)$$

Examination of equation (24) indicates that the procedure of replacing the residual by its least-squares fit is equivalent, for this problem, to solving the original problem with a modified Galerkin procedure in which the normal weight function,  $x^j$ , is replaced by  $x^{j-1}$ .

#### 4.2 Finite element solution of $dy/dx - y = 0$ with one element

The equation  $dy/dx - y = 0$  will be solved, in the region  $-1 \leq x \leq 1$  with the boundary condition  $y = e^{-1}$  at  $x = -1$ , using a Galerkin finite element formulation. The whole domain is spanned by one quadratic element and a solution in terms of the equally spaced nodal values,  $\bar{y}_1, \bar{y}_2$  and  $\bar{y}_3$ , is sought with  $\bar{y}_1$  known.

Substituting the quadratic finite element representation for  $y$  into the governing equation produces the residual

$$R = -\frac{1}{2} (x^2 - 3x + 1) \cdot \bar{y}_1 + (x^2 - 2x - 1) \cdot \bar{y}_2 - \frac{1}{2} (x^2 - x - 1) \cdot \bar{y}_3. \quad (25)$$



Application of a conventional Galerkin finite element formulation i.e.

$$\int_{-1}^1 N_i \cdot R \cdot dx = 0, \quad i = 2, 3. \quad (26)$$

produces the solution  $\bar{y}_2 = 1.75 \bar{y}_1$  and  $\bar{y}_2 = 5 \bar{y}_1$ . A least-squares fit of equation (25) produces the result

$$R_{1.s.} = \left(\frac{3}{2}x - \frac{2}{3}\right) \cdot \bar{y}_1 - \left(2x + \frac{2}{3}\right) \cdot \bar{y}_2 + \left(\frac{x}{2} + \frac{1}{3}\right) \cdot \bar{y}_3. \quad (27)$$

Requiring that  $R_{1.s.}$  is identically zero produces the solution  $\bar{y}_2 = 2.5 \bar{y}_1$  and  $\bar{y}_3 = 7 \cdot \bar{y}_1$ . The various solutions are compared with the exact solution in Table 2.

TABLE 2. SINGLE ELEMENT SOLUTIONS FOR  $dy/dx - y = 0$

x	Approximate solution		Exact solution $y = e^x$
	$y_{Gal.}$	$y_{res. l.s.}$	
-1	0.3679	0.3679	0.3679
0	0.6438	0.9197	1.0000
1	1.8394	2.5752	2.7183

If the integration in the conventional Galerkin finite element formulation had been carried out numerically a three point Gauss quadrature formula would have been required for the integration to be exact. If reduced integration had been used (i.e. a two point Gauss quadrature formula) the solution would have been identical with that produced by a least-squares fit of the residual. This is because the application of reduced integration to this problem is equivalent to replacing the residual by its least-squares fit and either using the original weight function or its least-squares fit. Since equation (27) is only linear in  $x$  and contains two unknown coefficients, the same solution will be obtained whatever weight function is used.

#### 4.3 Two element solution of $dy/dx - y = 0$

If the domain is not represented by a single element the solution is a little more complicated. The equation,  $dy/dx - y = 0$ , will be solved in the region  $0 \leq x \leq 1$  subject to the boundary condition  $y = 1$  at  $x = 0$ . The region is split into two elements : element A is  $0 \leq x \leq 0.5$  and element B is  $0.5 \leq x \leq 1$ . The solution will be obtained in terms of five, equally spaced, nodal values  $\bar{y}_1$  to  $\bar{y}_5$  with  $\bar{y}_1 = 1$ . The shape functions in elements A and B are quadratic in  $x$ .

Substitution of the finite element representation for  $y$  into the governing equation produces the following expressions for the residual,  $R$ .

In element A,

$$R = (-8x^2 + 22x - 7) \cdot \bar{y}_1 + (16x^2 - 40x + 8) \cdot \bar{y}_2 + (-8x^2 + 18x - 2) \cdot \bar{y}_3. \quad (28a)$$

In element B,

$$R = (-8x^2 + 30x - 20) \cdot \bar{y}_3 + (16x^2 - 56x + 32) \cdot \bar{y}_4 + (-8x^2 + 26x - 13) \cdot \bar{y}_5. \quad (28b)$$

Application of a conventional Galerkin formulation produces the results shown in Table 3 under  $\bar{y}_{ei}$ .

To apply the method developed in this paper it is necessary to fit the residuals, in the least-squares sense, separately in each element. This produces the result:

In element A,

$$R_{1.s.} = (18x - 6.6667) \cdot \bar{y}_1 + (-32x + 7.3333) \cdot \bar{y}_2 + (14x - 1.6667) \cdot \bar{y}_3. \quad (29a)$$

In element B,

$$R_{1.s.} = (18x - 15.6667) \cdot \bar{y}_3 + (-32x + 23.3333) \cdot \bar{y}_4 + (14x - 8.3333) \cdot \bar{y}_5. \quad (29b)$$

By requiring that  $R_{1.s.}$  in both elements are identically zero it is possible to obtain four relationships for the four unknowns:  $\bar{y}_2, \bar{y}_3, \bar{y}_4$  and  $\bar{y}_5$ . The results are shown in Table 3 under  $\bar{y}_{1s}$ . If a conventional method of weighted residuals were applied to  $R_{1.s.}$ , given by equations (29a) and (29b), the results for  $\bar{y}_2$  etc. would not differ from  $\bar{y}_{1s}$  shown in Table 3.

To produce the conventional Galerkin finite element solution, shown in Table 3, by performing the integration numerically, would require a three point Gauss quadrature formula. If a two point Gauss quadrature formula (reduced integration) is used to perform the numerical integration the result is as shown in Table 3 under  $y_r$ .

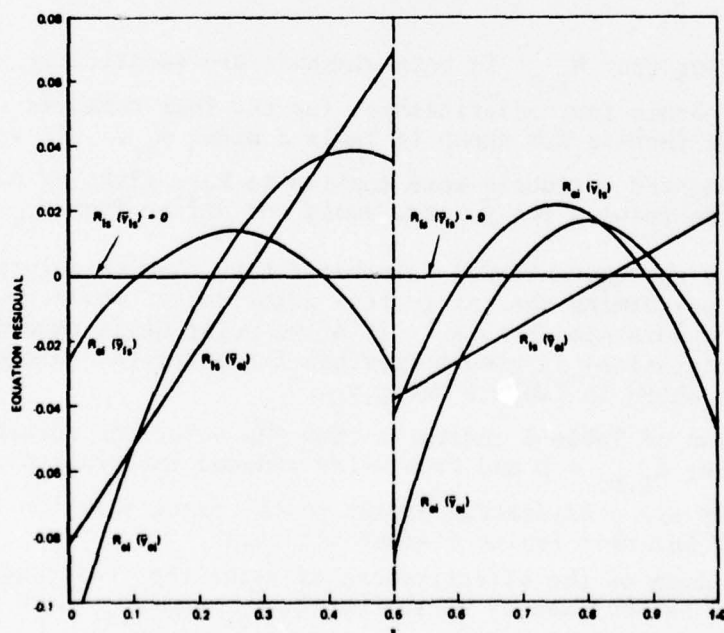
Examination of Table 3 indicates that the solutions obtained directly from requiring  $R_{1.s.} = 0$  and from using reduced integration are identical and that they are considerably closer to the exact solution than is the conventional Galerkin finite element solution.

An indication of the effectiveness of using the least-squares fit to the residual may be obtained by substituting,  $\bar{y}_{ei}$  and  $\bar{y}_{1s}$  shown in Table 3, into the expressions for the residuals, equations (28) and (29). The various combinations are shown in Figure 1.  $\int_0^1 R_{ei}$  refers to equations (28).

It is interesting to observe that  $R_{ei}(\bar{y}_{1s})$  is generally smaller than  $R_{ei}(\bar{y}_{ei})$  and is better distributed. This is also confirmed by the values of  $\int_0^1 R^2 dx$  shown in Table 3.

TABLE 3. TWO ELEMENT SOLUTIONS FOR  $dy/dx - y = 0$ 

x	Approximate solution (nodal), $\bar{y}$			Exact solution $y = e^x$
	Exact numerical integration $\bar{y}_{ei}$	Element least-squares fit of residual, $\bar{y}_{1s}$	Reduced numerical integration $\bar{y}_{ri}$	
0	1.0000	1.0000	1.0000	1.0000
0.25	1.2707	1.2838	1.2838	1.2840
0.50	1.6403	1.6486	1.6486	1.6487
0.75	2.0990	2.1165	2.1165	2.1170
1.00	2.6938	2.7180	2.7180	2.7183
$\int_0^1 R^2 dx$	0.00142	0.00021	0.00021	0.

Figure 1. Variation of equation residuals for  $dy/dx - y = 0$



#### 4.4 Steady viscous flow between parallel plates

The application of the present method in two dimensions is not quite so straightforward. Steady viscous flow between parallel plates has been used previously (ref. 12) to illustrate the Galerkin finite element formulation. The governing equation for this problem can be reduced to a Poisson equation. If the boundary conditions are chosen so that the problem has an exact solution the following specification can be obtained.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-\pi/2} \cdot \frac{\pi^2}{4} \cdot \cos \frac{\pi}{2} \cdot y \quad (30)$$

subject to the boundary conditions

$$u = 0 \text{ on } y = \pm 1 \text{ and } x = 1$$

$$u = (1 - e^{-\pi/2}) \cos \frac{\pi}{2} y \text{ on } x = 0. \quad (31)$$

The exact solution is  $u = (e^{-\pi x/2} - e^{-\pi/2}) \cos \frac{\pi}{2} y$ .  $u$  is a modified horizontal velocity difference.

A related problem will be considered here because it leads more directly to the problem considered in example 5. Equation (30) can be replaced by the two first order equations,

$$\frac{\partial r}{\partial x} + \frac{\partial s}{\partial y} = e^{-\pi/2} \cdot \frac{\pi^2}{4} \cdot \cos \frac{\pi}{2} \cdot y \quad (32a)$$

and

$$\frac{\partial r}{\partial y} - \frac{\partial s}{\partial x} = 0, \quad (32b)$$

where  $r = \partial u / \partial x$  and  $s = \partial u / \partial y$ . A solution will be sought in the domain  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$  subject to the boundary conditions,

$$\begin{aligned} r &= \frac{-\pi}{2} \cdot \cos \frac{\pi}{2} \cdot y \text{ on } x = 0 \\ r &= 0 \quad \text{on } y = 1 \\ s &= 0 \quad \text{on } x = 1 \text{ and } y = 0. \end{aligned} \quad (33)$$

One rectangular, quadratic Serendipity element is introduced to cover the domain, as indicated in Figure 2, and  $r$  and  $s$  are given a conventional finite element representation.



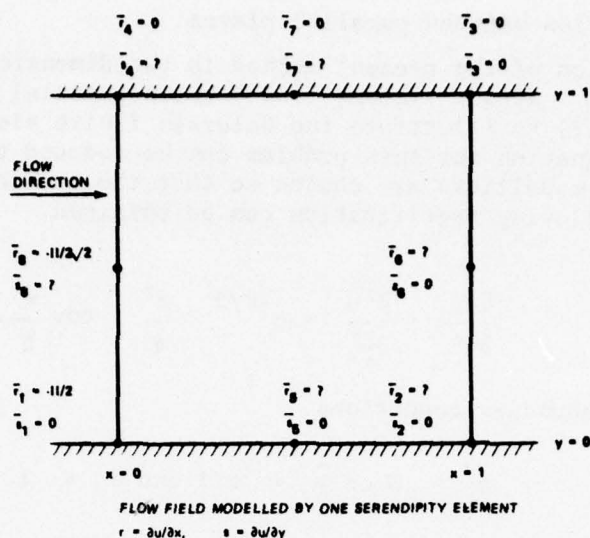


Figure 2. Viscous flow between two parallel plates

This problem has six nodal unknowns and Galerkin equations based on equations (32) are formed as follows

$$\int_0^1 \int_0^1 N_i \left\{ \sum_{j=1}^8 \frac{\partial N_j}{\partial x} \cdot \bar{r}_j + \sum_{j=1}^8 \frac{\partial N_j}{\partial y} \cdot \bar{s}_j - e^{-\pi/2} \cdot \frac{\pi^2}{4} \cdot \cos \frac{\pi}{2} \cdot y \right\} dx dy = 0$$

(34)

$i = 2, 5, 6$

and

$$\int_0^1 \int_0^1 N_i \left\{ \sum_{j=1}^8 \frac{\partial N_j}{\partial y} \cdot \bar{r}_j - \sum_{j=1}^8 \frac{\partial N_j}{\partial x} \cdot \bar{s}_j \right\} dx dy = 0, \quad i = 4, 7, 8.$$

(35)

The integrations are evaluated numerically; a 3 x 3 Gauss quadrature formula produces exact integration. In Table 4 are shown results for exact numerical integration and reduced numerical integration (2 x 2 Gauss quadrature formula).

The residuals in equations (34) and (35) have been fitted in the least-squares sense with representations of the form

$$R_{1s} = a_0 + a_1 x + a_2 y + a_3 \cdot x \cdot y.$$

(36)

Since each equation is characterised by four parameters which depend on the six nodal unknowns it is not possible to choose the nodal unknowns in such a way that  $R_{1s}$  are identically zero for both equations throughout the domain.

Consequently a Galerkin formulation of the form,

$$\int_0^1 \int_0^1 N_i \cdot R_{ls} \cdot dx dy = 0 \quad (37)$$

has been applied to each equation. The subsequent solution is shown in Table 4. It can be seen that the solutions obtained from a least-squares fit of the residual and from the use of reduced integration are of comparable accuracy and both are more accurate than the use of exact integration.

TABLE 4. SOLUTION FOR VISCOUS FLOW BETWEEN PARALLEL PLATES

Nodal parameter	Approximate solutions			Exact solution
	Exact numerical integration	Reduced numerical integration	L.S. fit of the residual	
$\bar{r}_2$	0.4103	0.3698	0.3728	0.3265
$\bar{r}_5$	0.7409	0.6788	0.6800	0.7162
$\bar{r}_6$	0.2621	0.2172	0.2172	0.2390
$\bar{s}_4$	1.5068	1.4234	1.4194	1.2443
$\bar{s}_7$	0.4313	0.4133	0.4114	0.3897
$\bar{s}_8$	0.8151	0.8100	0.8123	0.8798

The application of the Galerkin method results in algebraic equations of the form

$$[K] [q] = [B], \quad (38)$$

where  $[K]$  is the stiffness matrix and  $[q]$  is the vector of nodal unknowns. The stiffness matrices of the reduced integration formulation and the least-squares fit of the residual formulation are identical; however the matrices  $[B]$  are slightly different.

A typical term in the stiffness matrix is given by

$$k_{ij} = \int_0^1 \int_0^1 N_i \cdot \frac{\partial N_j}{\partial x} \cdot dx dy. \quad (39)$$

Both  $N_i$  and  $\partial N_j / \partial x$  are quadratic in  $x$  and  $y$  and therefore the product will be integrated exactly by a 3 x 3 Gauss quadrature formula. If a bilinear least-squares fit of  $\partial N_j / \partial x$  is made it will coincide with the values of  $\partial N_j / \partial x$  at the sampling points of a 2 x 2 Gauss quadrature formula(ref.14). The product  $N_i$  and  $(\partial N_j / \partial x)_{ls}$  is bicubic for which a 2 x 2 Gauss quadrature formula produces exact integration. Therefore

$$\int_0^1 \int_0^1 \underset{\text{reduced}}{N_i} \cdot \frac{\partial N_i}{\partial x} \cdot dx dy = \int_0^1 \int_0^1 N_i \cdot \left( \frac{\partial N_i}{\partial x} \right)_{1s} \cdot dx dy \quad (40)$$

A contributing term to [ B ] is the integration of

$$\int_0^1 \int_0^1 N_i \cdot \cos \frac{\pi}{2} y \cdot dx dy \quad (41)$$

Clearly a least-squares fit of  $\cos \frac{\pi}{2} y$  of the form given by equation (36) will not necessarily coincide with the values of  $\cos \frac{\pi}{2} y$  at the sampling points of a 2 x 2 Gauss quadrature formula. If the right hand side of the governing equation (32a) were a polynomial of second order or less then the solutions from the least-squares fit of the residual and from the use of reduced integration would be identical.

#### 4.5 Incompressible, inviscid flow

The final example to be considered is that of incompressible, inviscid flow past a two-dimensional circular cylinder; the flow-field is shown in Figure 3.

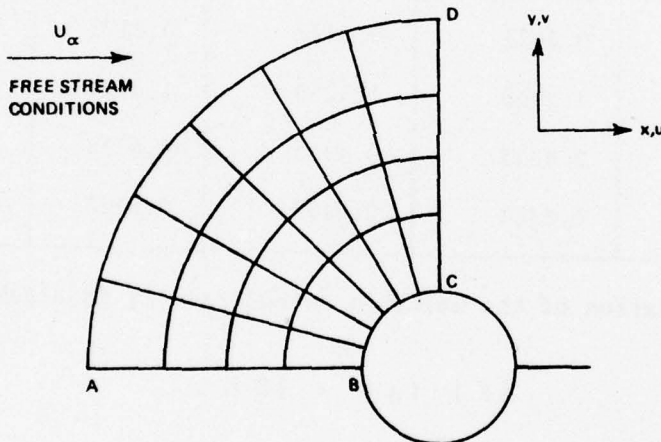


Figure 3. Flow field geometry for two-dimensional inviscid, incompressible flow

This example is more complex than those considered previously because the flow-field is represented by many elements and an isoparametric formulation is used to fit the curved nature of the body. However this problem, like the previous examples, possesses an exact solution so that direct comparison of various formulations is possible. This problem has been used previously to obtain a systematic comparison of various elements and shape functions for both exact and reduced integration (refs. 9, 15).

Results will be presented here that compare the least-squares fitting of the residual with exact and reduced integration for quadratic rectangular and triangular elements and two alternative Galerkin formulations.

The governing equations, for inviscid incompressible flow in two dimensions, are taken to be

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (42)$$



and

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0. \quad (43)$$

The Galerkin finite element formulation proceeds by introducing the following representation for u and v

$$u = \sum_j N_j \cdot \bar{u}_j \quad (44a)$$

and

$$v = \sum_j N_j \cdot \bar{v}_j, \quad (44b)$$

and requiring that

$$\iint N_i \cdot R_k \cdot dx \, dy = 0, \quad k = 1, 2 \text{ and } i = 1, n. \quad (45)$$

In equations (44) and (45),

$N_i$  is the shape function at the  $i^{\text{th}}$  node,

$\bar{u}_j, \bar{v}_j$  are the nodal values of the velocity components, u and v, and n is the total number of active nodes.

The residuals,  $R_k$ , are obtained by substituting equations (44) into equations (42) and (43), the result is

$$R_1 = \sum_j \frac{\partial N_j}{\partial x} \cdot \bar{u}_j + \sum_j \frac{\partial N_j}{\partial y} \cdot \bar{v}_j \quad (46a)$$

and

$$R_2 = \sum_j \frac{\partial N_j}{\partial y} \cdot \bar{u}_j - \sum_j \frac{\partial N_j}{\partial x} \cdot \bar{v}_j. \quad (46b)$$

Substitution of equations (46) into equations (45) and rearrangement gives

$$\sum_j a_{ij} \cdot \bar{u}_j + \sum_j b_{ij} \cdot \bar{v}_j = 0, \quad i = 1, n \quad (47)$$

and

$$\sum_j b_{ij} \cdot \bar{u}_j - \sum_j a_{ij} \cdot \bar{v}_j = 0, \quad i = 1, n. \quad (48)$$

A fuller description of the derivation of equations (47) and (48) is given elsewhere(ref.15).



The first set of results to be presented have been obtained with quadratic, Serendipity, rectangular elements. The form of  $a_{ij}$  and  $b_{ij}$  in equations (47) and (48) depend on whether Green's theorem has been applied. Two cases will be considered:

Case 1: Green's theorem applied to the Galerkin equations

$$a_{ij} = \iint \frac{\partial N_i}{\partial x} \cdot N_j \cdot dx \, dy - \int N_i \cdot N_j \cdot l_x \cdot ds \quad (49)$$

and

$$b_{ij} = \iint \frac{\partial N_i}{\partial y} \cdot N_j \cdot dx \, dy - \int N_i \cdot N_j \cdot l_y \cdot ds. \quad (50)$$

The line integrals can only contribute if the  $i^{\text{th}}$  node lies on the boundary and even then may produce zero contribution due to the boundary conditions or the values of the direction cosines,  $l_x$  and  $l_y$ . For an isoparametric formulation used with rectangular elements the integrations in equations (49) and (50) are carried out in the plane of a regular rectangular element based on a  $\xi, \eta$  coordinate system. The result is, for the area integrals,

$$a_{ij} = \iint N_j \left\{ \sum_{k=1}^8 \left( \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right) \cdot y_k \right\} d\xi \cdot d\eta \quad (51)$$

and

$$b_{ij} = \iint N_j \left\{ \sum_{k=1}^8 \left( \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right) \cdot x_k \right\} d\xi \cdot d\eta. \quad (52)$$

$x_k, y_k$  are the coordinates of the  $k^{\text{th}}$  node in the element.

Case 2: Green's theorem not applied to the Galerkin equations

$$a_{ij} = \iint N_i \cdot \frac{\partial N_j}{\partial x} \, dx \, dy \quad (53)$$

$$b_{ij} = \iint N_i \cdot \frac{\partial N_j}{\partial y} \, dx \, dy. \quad (54)$$

For an isoparametric formulation with rectangular elements equations (53) and (54) become

$$a_{ij} = \iint N_i \left\{ \sum_{k=1}^8 \left( \frac{\partial N_j}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_j}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right) \cdot y_k \right\} d\xi \cdot d\eta \quad (55)$$

and

$$b_{ij} = - \iint N_i \sum_{k=1}^8 \left( \frac{\partial N_j}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_j}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right) \cdot x_k \, d\xi \cdot d\eta. \quad (56)$$

In equations (51), (52), (55) and (56) it is possible to identify those parts of the equation that come from the residual and those parts that come from the weight function in equation (45). Once the element and shape function are chosen it is possible to deduce the order of the contributions to the weight function and residual. The results of this are shown in Table 5 and will be made use of in the discussion of the solutions.

TABLE 5. ORIGIN AND ORDER OF EQUATIONS (51), (52), (55), (56) and (60) TO (63)

Element type	Green's theorem applied	Weight function		Residual	
		Terms from	Order of	Terms from	Order of
Rect-angular quadratic element	YES	$\frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi}$	bi-cubic	$N_j$	bi-quadratic
	NO	$N_i$	bi-quadratic	$\frac{\partial N_j}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_j}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi}$	bi-cubic
Rect-angular quadratic element	YES	$\frac{\partial N_i}{\partial L_1} \cdot \frac{\partial N_k}{\partial L_2} - \frac{\partial N_i}{\partial L_2} \cdot \frac{\partial N_k}{\partial L_1}$	bi-quadratic	$N_j$	bi-quadratic
	NO	$N_i$	bi-quadratic	$\frac{\partial N_j}{\partial L_1} \cdot \frac{\partial N_k}{\partial L_2} - \frac{\partial N_j}{\partial L_2} \cdot \frac{\partial N_k}{\partial L_1}$	bi-quadratic

The computational solutions, considered under Example 5, have been obtained within the region ABCD in Figure 3. The nodal points and the elements have been defined on a polar grid and an isoparametric formulation has been used to connect this to a cartesian grid. All results presented are for the variation, with angular position, of the tangential velocity component at the body surface.

A root mean square difference,  $\sigma$ , between the finite element solution and the exact solution at the body surface is defined as follows

$$\sigma = \left[ \left\{ \sum_{i=1}^{N_B} (q_T - q_e)^2 \right\} / N_B \right]^{1/2}, \quad (57)$$

where  $q_T$  is the finite element solution for the tangential velocity component,  $q_e$  is the exact solution for the tangential velocity component and  $N_B$  is the number of nodes between  $\theta = 0$  and  $90^\circ$  (B and C in Figure 3).  $\sigma$  has been found useful for comparing results in tabulated form. A summary of the results, for the various cases considered under Example 5, is given in Table 6.

TABLE 6. SUMMARY OF SOLUTIONS FOR INVISCID, INCOMPRESSIBLE FLOW ABOUT A CIRCULAR CYLINDER

Element type	Green's theorem applied	Number of elements	Number of unknowns	Nodal R.M.S. differences		
				Exact numerical integration	Exact numerical integration	Least-squares fit of residual
Quadratic rectangular (Serendipity)	YES	25	149	0.049	0.015	0.040
	NO	25	149	0.062	-	0.087
Quadratic rectangular	YES	50	199	0.126	0.063	0.059
	NO	50	199	0.252	0.481	0.018

The first group of results were obtained with 25 quadratic, Serendipity, rectangular elements spanning the flow-field. This required 149 nodal unknowns to be solved for. For case 1, in which Green's theorem is applied, the results are shown in Figure 4. Results have been obtained for exact numerical integration, reduced numerical integration and for a bilinear least-squares fit of the residual.

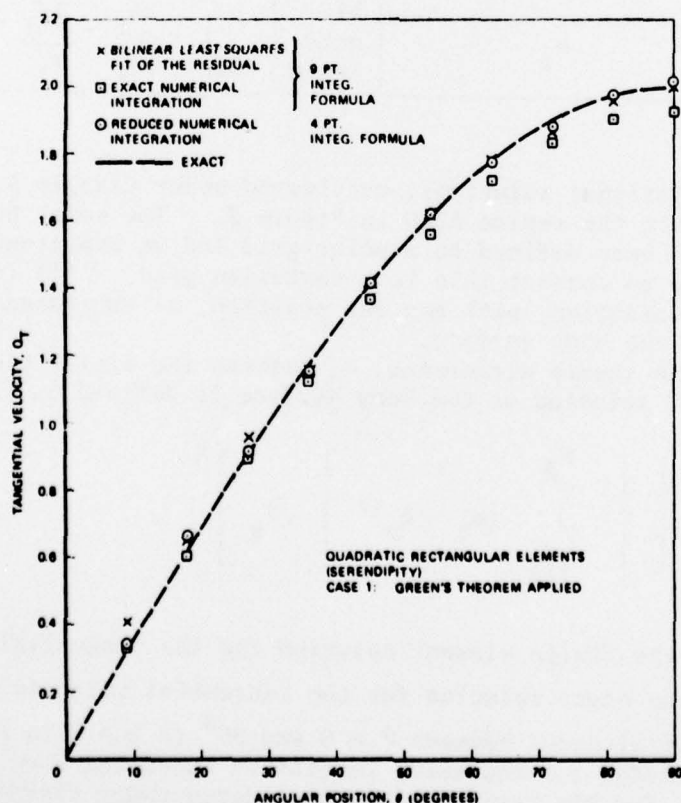


Figure 4. Comparison of surface velocities - rectangular elements - Green's theorem applied



It is apparent that the results for reduced integration are better than those produced by a least-squares fit of the residual, and that both are better than the results using exact numerical integration. In obtaining the solution using reduced integration, integrals of the form

$$\iint N_j \left\{ \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right\} d\xi \cdot d\eta \quad (58)$$

are evaluated using a 2 x 2 Gauss quadrature formula. A 2 x 2 Gauss quadrature formula is capable of integrating a bicubic integrand exactly; the integrand in expression (58) is biquintic. Referring to Table 5 it can be seen that  $N_j$ , which comes from the residual is biquadratic and the term  $\left\{ \right\}$ , which comes from an isoparametric transformation of the weight function, is bicubic.

Sampling  $N_j$  at the second order Gauss points is equivalent to replacing  $N_j$  by a bilinear least-squares fit of  $N_j$  (ref.14). Therefore sampling the term  $\left\{ \right\}$  in expression (58) at the second-order Gauss points may be interpreted as fitting the term with a biquadratic or bilinear function that matches the original function at the sampling points.

In contrast with this the 'least-squares fit of the residual' results have replaced  $N_j$  in expression (58) by a least-squares fit but have left the term  $\left\{ \right\}$ , that comes from the weight function, in its original form.

This has necessitated the use of a 3 x 3 Gauss quadrature formula. Thus the differences in the reduced integration solution and the "least-squares fit of the residual" solution arise from the different treatment of the weight function. Why the reduced integration treatment of the weight function should produce superior results is not clear.

For case 2, in which Green's theorem is not applied to the Galerkin equations, the results are shown in Figure 5. The use of reduced integration failed to produce a result and the least-squares fit of the residual has produced a result that is inferior to that produced by exact numerical integration.

For this case it is necessary to carry out integrations of the form

$$\iint N_i \left\{ \frac{\partial N_j}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_j}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right\} \cdot d\xi \cdot d\eta \quad (59)$$

Reference to Table 5 indicates that the term  $\left\{ \right\}$  in expression (59) contributes to the residual. This term is bicubic and therefore a least-squares fit of this term will need to be biquadratic. However this requires nine unknown coefficients which is as many as is required to define the overall integration exactly. In terms of the required inequality (7) M is not reduced. Thus a biquadratic fit, in this case, violates the original requirement of reducing the number of constraints on the residual. The results in Figure 5 give some idea of the error in replacing the residual with its least-squares fit when no gain is obtained from a reduction in the number of constraints.

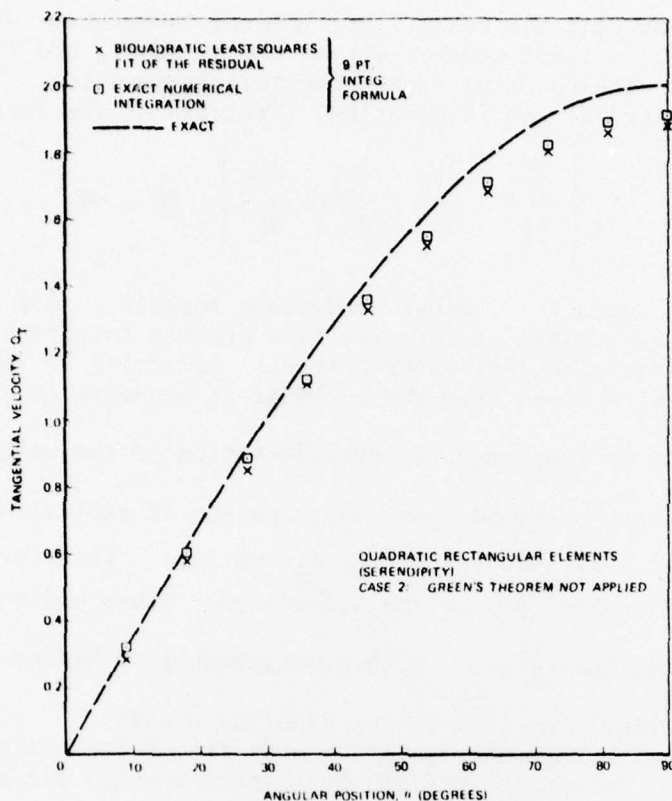


Figure 5. Comparison of surface velocities - rectangular elements - Green's theorem not applied

An attempt at a reduced integration solution implies an attempt to replace the bicubic residual term with either a bilinear or biquadratic fit of the residual term whose values at the  $2 \times 2$  Gauss points match the original residual term. Clearly neither possibility has the required least-squares property of minimising the square of the residual.

The second group of results were obtained with 50 quadratic triangular elements spanning the flow-field. A solution has been sought for 199 nodal unknowns. Once an isoparametric transformation, in terms of the triangular coordinates  $L_1, L_2$ , has been applied to equations (49), (50), (53) and (54) the relevant expressions for  $a_{ij}$  and  $b_{ij}$  are:

Case 1: Green's theorem applied to the Galerkin equations

$$a_{ij} = \iint N_j \left\{ \sum_{k=1}^8 \left( \frac{\partial N_i}{\partial L_1} \cdot \frac{\partial N_k}{\partial L_2} - \frac{\partial N_i}{\partial L_2} \cdot \frac{\partial N_k}{\partial L_1} \right) \cdot y_k \right\} dL_1 \cdot dL_2 \quad (60)$$

and

$$b_{ij} = \iint N_j \left\{ \sum_{k=1}^8 \left( \frac{\partial N_i}{\partial L_1} \cdot \frac{\partial N_k}{\partial L_2} - \frac{\partial N_i}{\partial L_2} \cdot \frac{\partial N_k}{\partial L_1} \right) \cdot x_k \right\} dL_1 \cdot dL_2 \quad (61)$$

Case 2: Green's theorem not applied to the Galerkin equations

$$a_{ij} = \iint N_i \left\{ \sum_{k=1}^8 \left( \frac{\partial N_j}{\partial L_1} \cdot \frac{\partial N_k}{\partial L_2} - \frac{\partial N_j}{\partial L_2} \cdot \frac{\partial N_k}{\partial L_1} \right) \cdot y_k \right\} dL_1 \cdot dL_2 \quad (62)$$

and

$$b_{ij} = - \iint N_i \left\{ \sum_{k=1}^8 \left( \frac{\partial N_j}{\partial L_1} \cdot \frac{\partial N_k}{\partial L_2} - \frac{\partial N_j}{\partial L_2} \cdot \frac{\partial N_k}{\partial L_1} \right) \cdot x_k \right\} dL_1 \cdot dL_2. \quad (63)$$

In equations (60) to (63) it is possible to identify which parts of the equations come from the weight function and which parts come from the residual in equation (45). This information is given in Table 5 along with the order of the various terms in equations (60) to (63).

For case 1 above the results are shown in Figure 6. Results are presented for exact numerical integration, for reduced numerical integration and for a least-squares fit of the residual of the following form

$$R_{1s} = a_0 + a_1 \cdot L_1 + a_2 \cdot L_2. \quad (64)$$

A seven point formula was used to produce the exact numerical integration results and a four point formula, due to Cowper(ref.16), was used to produce the reduced integration results; these quadrature formulae are described elsewhere(ref.7).

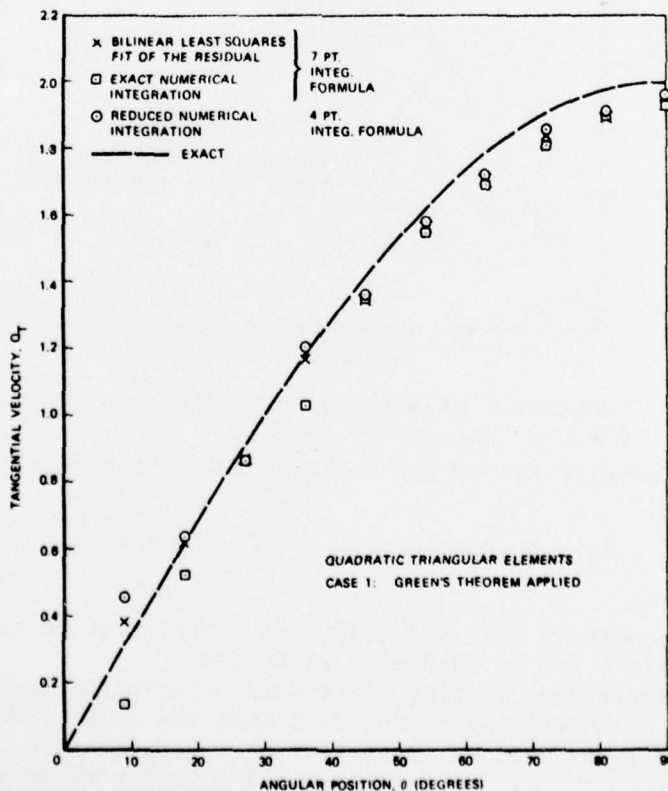


Figure 6. Comparison of surface velocities - triangular elements - Green's theorem applied



The "least-squares fit of the residual" results are better than the reduced integration results and both are better than the exact integration results (see Table 6 also). It is interesting that the reduced integration results have required four evaluations of the residual per element whereas the "least-squares fit of the residual" results have required only three parameters per element. Thus, in terms of the inequality (7), the number of unknowns  $n$  is 199 for both cases. The total number of function evaluations,  $M$ , for reduced integration is 400 and for the least-squares fit of the residual is equivalent to 300. Examination of Table 5 indicates that  $N_j$  in equation (60) and (61) is biquadratic so that a least-squares fit that is linear in  $L_1$  and  $L_2$  would appear appropriate.

The results for case 2, when Green's theorem is not applied, are shown in Figure 7. For this case the results using exact integration are poor and the results using reduced integration are worse. But the results using a least-squares fit of the residual are very good, virtually as good as the best reduced integration results obtained for this problem (Figure 4).

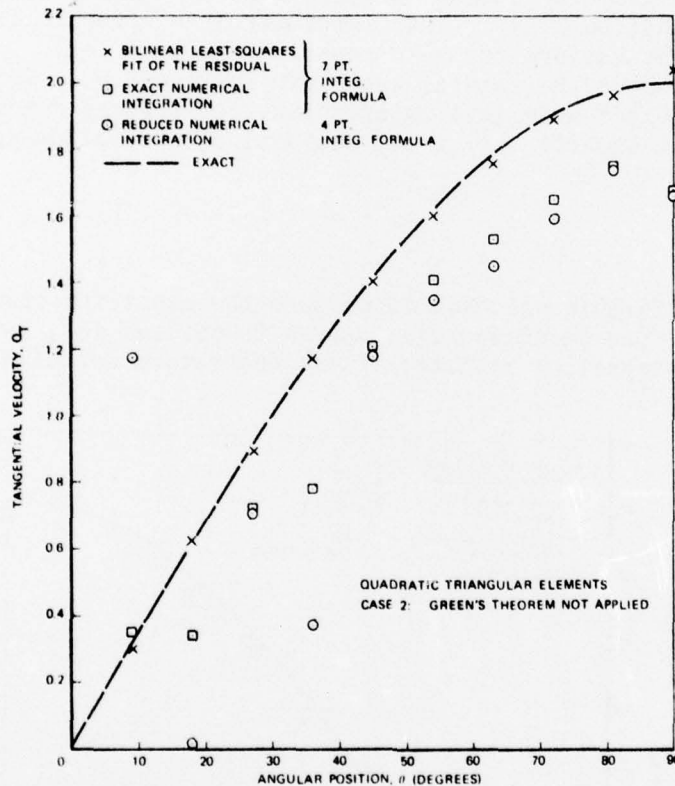


Figure 7. Comparison of surface velocities - triangular elements - Green's theorem not applied

The least-squares fit of the residual was of the form

$$R_{1s} = a_0 + a_1 \cdot L_1 + a_2 \cdot L_2 + a_3 \cdot L_1 \cdot L_2. \quad (65)$$

The term in equations (62) and (63) that contributes to the residual is listed in Table 5 and is quadratic in  $L_1$  and  $L_2$ .

The relatively poor showing of reduced integration applied to triangular elements may be attributed to the fact that the lower order numerical integration formula used has no least-squares interpretation as has the corresponding Gaussian quadrature formula(ref.14) used with rectangular elements.

## 5. DISCUSSION

Least-squares fitting has been used previously in finite element formulation. In the area of plate bending, Irons and Tazzaque(ref.17,18) have started with 9 and 12 degree of freedom triangular elements and replaced the second derivative of the shape function by its least-squares fit. This has allowed the use of a lower order numerical integration formula and produced superior results to the elements they started with.

Hinton and Campbell(ref.19) have used local and global smoothing in order to improve stresses obtained from numerically integrated, two dimensional, isoparametric elements. Hinton and Campbell note that, for rectangular, quadratic elements the evaluation of the stiffness integral

$$\iint B^T \cdot D \cdot B \cdot dA \quad (66)$$

by reduced integration (2 x 2 Gauss quadrature formula) produces the same result as performing the exact integration of

$$\iint \tilde{B}^T \cdot \tilde{D} \cdot \tilde{B} \cdot dA \quad (67)$$

where  $\sim$  indicates a local least-squares fit of that term. Since each least-squares fit is bilinear, the integrand of expression (67) is bicubic and can be integrated exactly by a 2 x 2 Gauss quadrature formula. The integrand is of the same order as that produced when expression (58) is replaced by

$$\iint N_{j1s} \left\{ \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial N_k}{\partial \eta} - \frac{\partial N_i}{\partial \eta} \cdot \frac{\partial N_k}{\partial \xi} \right\} d\xi \cdot d\eta. \quad (68)$$

It is apparent from previous applications, and from the results presented in this paper, that reduced integration has produced superior results to the use of exact integration for rectangular elements but has not been effective for triangular elements.

An important step in the formulation, presented here as an alternative to reduced integration, has been to recognise that if the residual is written

$$R = \sum_{j=1}^L R_j(x,y) \cdot \bar{q}_j \quad (69)$$

where  $\bar{q}_j$  are the nodal parameters and  $R_j(x,y)$  are made up of shape function derivatives, etc. determined by the governing equations, then it follows that

$$R_{1s} = \sum_{j=1}^L R_{j1s}(x,y) \cdot \bar{q}_j. \quad (70)$$

Equation (70) is particularly useful since it is straightforward to form  $R_{j1s}$  on an element basis. If the numerical integrations are performed on a dummy element, once and for all(ref.15), the formation and evaluation of  $R_{j1s}$  is also economical.

It has been demonstrated that the use of a modified method of weighted residuals

$$\iint W \cdot R_{1s} \cdot dA = 0 \quad (71)$$

has produced more accurate results than the use of

$$\iint W \cdot R \cdot dA = 0 \quad (72)$$

in a larger number of situations than has the use of reduced integration applied to equation (72). In particular the current formulation has extended the benefits associated with reduced integration to triangular elements.

It seems likely that, where reduced integration produces very accurate results, the increased accuracy over the use of equation (71) comes from the implicit treatment of the weight function,  $W$ . This would appear to be a fruitful area for future research.



## REFERENCES

- | No. | Author  | Title   |
|-----|---|---|
| 1.  | O.C. Zienkiewicz,<br>T.L. Taylor and<br>J.M. Too. | "Reduced integration techniques in<br>general analysis of plates and shells"<br>Int. J. Num. Meth. Engng., <u>3</u> , 275-290<br>(1971).  |
| 2.  | S.F. Pawsey and<br>R.W. Clough                    | "Improved numerical integration of thick<br>shell finite elements"<br>Int. J. Num. Meth. Engng., <u>3</u> , 575-586<br>(1971).  |
| 3.  | O.C. Zienkiewicz,<br>D.R.J. Owen and<br>K.N. Lee  | "Least-square finite element for elasto-<br>static problems. Use of 'reduced'<br>integration"<br>Int. J. Num. Meth. Engng., <u>8</u> , 341-358<br>(1974).   |
| 4.  | O.C. Zienkiewicz and<br>P.N. Godbole              | "Viscous, incompressible flow with<br>special reference to non-Newtonian<br>(plastic) fluids"<br>in <u>Finite Elements in Fluids</u> . Vol. 1,<br>ed. by R.H. Gallagher et al., Wiley,<br>25-55 1975.                       |
| 5.  | R.S. Barsoum                                      | "A degenerate solid element for linear<br>fracture analysis of plate bending and<br>general shells"<br>Int. J. Num. Meth. Engng., <u>10</u> , 551-564<br>(1976).  |
| 6.  | J.C. Heinrich and<br>O.C. Zienkiewicz             | "Quadratic finite element schemes for<br>two dimensional convective transport<br>problems"<br>Report C/R/284/76, Dept. of Civil Eng.,<br>University of Wales (Swansea), Oct. 1976.  |
| 7.  | C.A.J. Fletcher                                   | "Improved integration techniques for<br>fluid flow finite element formulation"<br>W.R.E. TR-1810(WR&D) Salisbury, S.A.,<br>April 1977.  |
| 8.  | J.H. Argyris and<br>K.J. Willam                   | "Some considerations for the evaluation<br>of finite element models"<br>Nuclear Eng. Design, <u>28</u> , 76-96 (1974).  |
| 9.  | O.C. Zienkiewicz and<br>E. Hinton                 | "Reduced integration, function smoothing<br>and nonconformity in finite element<br>analysis (with special reference to<br>thick plates)"<br>Proc. Inte. Conf. Fin. Elem. Meth.<br>Engng, University of Adelaide, Dec. 1976. |
| 10. | B.M. Irons  | "Finite element techniques in structural<br>mechanics"<br>Discussion H, ed. Tottenham and Brebbia,<br>Southampton U.P., 328-331, 1970.  |

No.	Author	Title
11.	B.A. Finlayson and L.E. Scriven	"The method of weighted residuals - a review" Applied Mechanics Reviews, <u>19</u> , 735-748, (1966).
12.	C.A.J. Fletcher	"The Galerkin Method : an introduction" Proc. 1st. Aust. Conf. Fluid Dynamic Simulation, Monash Univ., Melbourne publ. by North-Holland Publ. Co., Amsterdam, 1977.
13.	W.J. Duncan	"Galerkin's method in mechanics and differential equations" ARC R&M 1798, 1937.
14.	E. Isaacson and H.B. Keller	"Analysis of numerical methods" Wiley, New York, 1966.
15.	C.A.J. Fletcher	"The application of the finite element method to two-dimensional, inviscid flow" WRE-TN-1606(WR&D), Salisbury, S.A., May 1976.
16.	G.R. Cowper	"Gaussian quadrature formulas for triangles" Int. J. Num. Meth. Engng., <u>4</u> , 405-408 (1972).
17.	B.M. Irons and A. Razzaque	"Shape function formulations for elements other than displacement models" in <u>Variational Methods in Engineering</u> , <u>Vol.1.</u> ed. Brebbia and Tottenham, Southampton Univ. Press, 4/59-4/72, 1973.
18.	A. Razzaque	"Program for triangular bending elements with derivative smoothing" Int. J. Num. Meth. Engng., <u>6</u> , 333-343 (1973).
19.	E. Hinton and J.S. Campbell	"Local and global smoothing of dis- continuous finite elements functions using a least-squares method" Int. J. Num. Meth. Engng., <u>8</u> , 461-480 (1974).

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